Variational Techniques Applied to Capture in Phase-Controlled Oscillators

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A variational formulation for deriving bounds on the capture (pull-out) range of phase-controlled oscillators is developed. This technique is applied to the more common types of systems, viz., those involving symmetric comparators and simple lag-RC filters. Exact asymptotic capture expressions relating to the simple RC filters with small bandwidth are obtained.

I. INTRODUCTION

The synchronization behavior of phase-controlled oscillators has been for many years the subject of extensive analytic studies. Although yielding significant and experimentally consistent results, these studies have been based primarily on laborious graphical procedures and linearizing approximations. In this paper we discuss and apply a variational formulation with which to analyze the synchronization phenomena of phase-controlled systems more directly and somewhat more generally.

The phase-controlled oscillator is represented by the block diagram of Fig. 1(a). The basic function of such a system is to establish and maintain, within certain limits, an approximately zero difference in frequency between the input and local carriers; i.e., the local oscillator is constrained so as to follow variations in the frequency of the input carrier. As indicated, this control is effected by applying the two carriers to a phase comparator and linear filter in cascade, and utilizing the resultant output to appropriately control the frequency of the local oscillator. The comparator output in most cases is a simple function of the phase difference $\varphi_i - \varphi_o$ between the carriers, the type of function depending to a large extent on the particular physical operation by which the phase difference is measured. For example, if the frequency spectrum of $\varphi_i - \varphi_o$ is baseband, then multiplying the two carriers and filtering out the resultant high-frequency component yield an output which approximates that of an ideal "cosine" comparator, viz., $f(\varphi_i - \varphi_o) \sim \cos(\varphi_i - \varphi_o)$. In general,

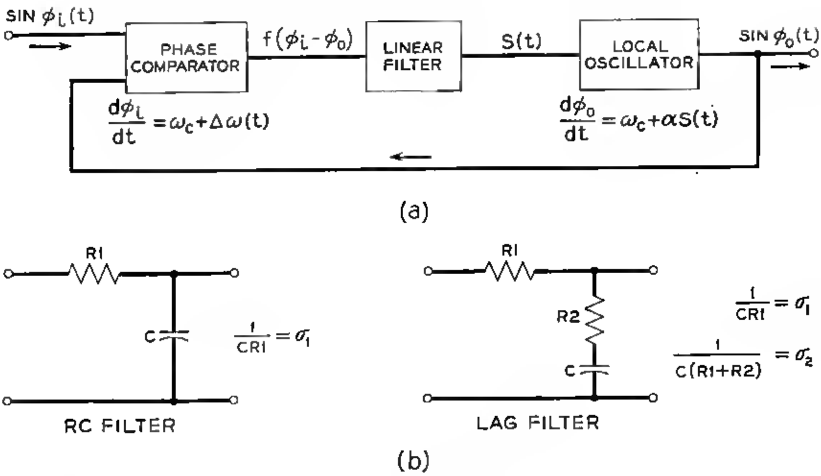

Fig. 1 — (a) Block diagram of the phase-controlled oscillator; (b) RC and lag filters.

$f(\varphi_i - \varphi_o)$ is not a baseband function; however, the filter depicted in Fig. 1(a) is usually a low-pass lag or RC type, serving to eliminate any high-frequency deviations in the control branch.

As applied here, the term synchronization relates to the difference in frequency between the input and output carriers; specifically, we consider the system to be in a synchronous state provided this frequency difference is either identically or asymptotically zero, i.e.,

$$(d\varphi_i/dt) - (d\varphi_o/dt) = 0 \quad \text{or} \quad \rightarrow \quad 0 \ (t \rightarrow \infty).$$

Despite the fact that these conditions are realized in special cases only, near-synchronous states are readily established by most practical systems. Whether an exact, near, or nonsynchronous state prevails in a given situation depends largely on the particular form of input $d\varphi_i/dt$ and the related initial conditions. If completely general, the analysis of such phenomena necessarily involves both arbitrary inputs and arbitrary initial conditions. Owing to the intractable nonlinear formulation encountered, however, a broad treatment of synchronization does not appear possible at present except for trivial systems.

To establish effective criteria for evaluating system performance yet obviate the mathematical difficulties associated with a more detailed analysis, we consider special classes of input functions that in cases of interest characterize actual inputs and lead to exact or approximate solu-

tions. The most acceptable as well as convenient input is perhaps the step function, a single "jump" in frequency of magnitude $C_1(t \geqq 0)$, normally applied with the phase difference initially zero and the system initially in a synchronous state. Such functions represent the more extreme or sudden type of input deviation, and yield in the simple lag-RC filter cases (cf. Fig. 1(b)) autonomous formulations to which phase plane descriptions and techniques are applicable. We often find it possible to consider inputs with less restrictive initial conditions, namely those satisfying the asymptotic relation

$$(d\varphi_i/dt) - (\omega_c) \to C_2 = \text{const} \ (t \to \infty).$$

(Here, ω_c represents the "free-running" frequency of the local oscillator and C_2, an asymptotic frequency jump.*) The maximum positive values of C_1 and C_2 for which the related functions restore the system to a synchronous state are defined as the positive "pull-out" (capture) and "pull-in" ranges, respectively. As measures of performance, these definitions have been used extensively, especially in cases involving sine comparators and lag-RC filters.

One essential purpose of this paper is to formulate a variational procedure whereby one can obtain analytically bounds on the capture range. The method applies specifically to the second order, autonomous, nonlinear equation,

$$\frac{d^2x}{dt^2} = F\left(x, \frac{dx}{dt}\right)$$

which for $x \sim \varphi_i - \varphi_o$ describes phase-controlled systems with lag-RC filters and step function inputs. Relative to the phase plane representation given in Fig. 2, solution trajectories of the above equation for lag-RC systems are of two principal types: a nonsynchronous form running on indefinitely, and a synchronous or capture form satisfying the con-

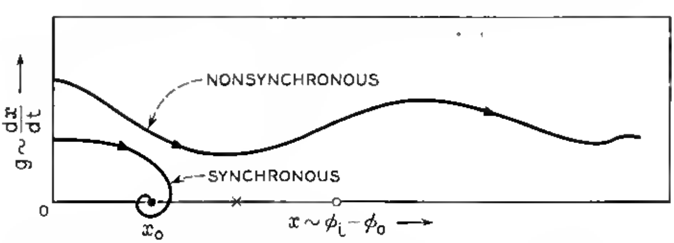

Fig. 2 — Phase plane representation of solution trajectories.

* See list of symbols, Appendix D.

dition $(d\varphi_i/dt) - (d\varphi_o/dt) \rightarrow 0$. As discussed previously, the basic problem is determining values of C_1 for which only the latter form results. To find these values by phase plane methods requires fairly accurate graphical plots of a large number of solution trajectories. Of main concern, however, is the relationship between C_1 and the resultant solution type, not detailed knowledge of the trajectory. Utilizing this principle, we show below that subintervals of C_1 can be identified as corresponding to either capture or noncapture trajectories by means of "test" trajectories, prescribed functions exhibiting the essential features of the actual solutions. The more closely the test contours fit those of the solutions, the more closely the derived subintervals bound the capture and noncapture ranges. For obtaining optimal bounds, we normally construct a class of appropriate test functions and vary over that class, the procedure simulating to some extent that of linear vector-space theory where variational formulations based on trial eigenfunctions furnish bounds on eigenvalues. Although the development is restricted to the second-order, autonomous equation given above, straightforward generalizations enable one to deal with higher-order, nonautonomous equations.

In the treatment that follows, we consider first the mathematical formulation of phase-controlled systems involving lag-RC filters, then the variational formulation, and finally the application of the latter to symmetric comparator functions, i. e.,

$$\begin{aligned} f(\epsilon) &= -f(-\epsilon) \quad (\epsilon = \varphi_i - \varphi_o) \\ f(\epsilon) &= f(\epsilon + 2\pi) \\ f(\epsilon) &\geqq 0 \quad (0 \leqq \epsilon \leqq \pi). \end{aligned}$$

Regarding the symmetric comparator and RC filter case, several results are derived: (*i*) bounds on the capture range under general bandwidth conditions; (*ii*) exact asymptotic formulae for the capture range under small and large bandwidth conditions; (*iii*) the fact that for capture solutions, the steady-state phase error lies in the interval

$$-\pi \leqq \epsilon \leqq \pi.$$

With respect to sawtooth comparator functions, i. e.,

$$f(\epsilon) = \sum_{n=-\infty}^{\infty} f_0(\epsilon - 2\pi n)$$

$$f_0(\epsilon) = \begin{cases} \epsilon & (|\epsilon| < \pi) \\ 0 & (|\epsilon| \geqq \pi) \end{cases}$$

we determine exact capture relations for RC filters and capture bounds for lag filters.

II. GENERAL FORMULATION

As in previous investigations,[1,2] the mathematical representation of the phase-controlled system shown in Fig. 1 is based primarily on three physical constraints:

(i) With regard to the control of the local oscillator, the instantaneous frequency $\omega_0(t)$ of the local carrier consists of a constant, free-running component ω_c which is independent of the control signal $S(t)$ applied to the local oscillator, and a component which is directly proportional to $S(t)$; i. e.,

$$\omega_0(t) = \frac{d\varphi_0}{dt} = \omega_c + \alpha S(t). \tag{1}$$

In general, ω_c is chosen so as to lie within the frequency range of the input signal. Accordingly, we represent the input as the sum of two frequency components, ω_c and a time-dependent deviation $\Delta\omega(t)$; viz.,

$$\omega_i(t) = \frac{d\varphi_i}{dt} = \omega_c + \Delta\omega(t). \tag{2}$$

(ii) Regarding synchronism, it is required that the system exhibit a "natural" synchronous state for which $\omega_i(t) = \omega_0(t) = \omega_c(t \geqq 0)$ provided only $\epsilon(0) = \varphi_i(0) - \varphi_o(0) = 0$ and $\Delta\omega(t) = 0$ $(t \geqq 0)$. If, after being initially established, this state is to persist indefinitely, then $S(t) = 0$ $(t \geqq 0)$. The latter condition requires the comparator output to vanish for zero error input and the linear filter to be initially inert. More explicitly, it is stipulated that

$$S(t) = 0 \quad (\epsilon(t) = 0, \qquad t \geqq 0) \tag{3}$$

$$f(0) = 0 \tag{4}$$

$$\left(\frac{df}{d\epsilon}\right)_{\epsilon=0} \leqq m = \text{const} < \infty. \tag{5}$$

In addition, the usual supposition of physical realizability implies the requisite

$$|f(\epsilon)| \leqq f_m = \text{const} < \infty \quad (-\infty < \epsilon < \infty). \tag{6}$$

(iii) Relative to boundary conditions, we assume that with the system initially in the synchronous state outlined in (ii), there is applied at

$t = 0$ an arbitrary $\Delta\omega(t)$ depending only on the behavior to be analyzed:

$$\epsilon(0^-) = 0, \qquad \Delta\omega(t) = 0 \quad (t < 0). \tag{7}$$

The constitutive conditions given by (1) through (7) yield

$$\frac{d\epsilon}{dt} = \Delta\omega(t) - \alpha \int_0^t H(t-\tau) f[\epsilon(\tau)]\, d\tau \quad (t \geqq 0) \tag{8}$$

where the filter impulse response $H(t)$, $\Delta\omega(t)$, and $f(\epsilon)$ are regarded as generalized functions.[3] For physically realizable filters and comparators, the convolution in the last term of this expression is bounded in a neighborhood of the point $t = 0$; hence, provided $\Delta\omega(t)$ is bounded, $d\epsilon/dt$ behaves similarly in this neighborhood and

$$\epsilon(0^-) = \epsilon(0) = \epsilon(0^+) = 0. \tag{9}$$

Considering specifically the lag filter for which

$$H(t) = \mathcal{L}^{-1}\left[\frac{\sigma_1}{\sigma_2}\left(\frac{s+\sigma_2}{s+\sigma_1}\right)\right] = \frac{\sigma_1}{\sigma_2}[\delta(t) + (\sigma_2 - \sigma_1)e^{-\sigma_1 t}] \quad (t \geqq 0) \tag{10}$$

and differentiating (8) with respect to t, one obtains

$$\begin{aligned} \frac{k}{\sigma_1}\frac{dg}{dt} = \frac{1}{\alpha f_m}&\left(\Delta\omega + \frac{1}{\sigma_1}\frac{d\Delta\omega}{dt}\right) \\ &- C(x) - kg + (1-\beta)kg\frac{dC}{dx} \quad (t > 0) \end{aligned} \tag{11}$$

$$\frac{dx}{dt} = \frac{\sigma_1}{k} g \tag{12}$$

where

$$x(t) = \frac{1}{\pi}\epsilon(t), \qquad C(x) = \frac{1}{f_m} f(\pi x)$$

$$\beta = 1 + \frac{\alpha f_m}{\pi\sigma_2}, \qquad k = \left(\frac{\pi\sigma_1}{\alpha f_m}\right)^{\frac{1}{2}}.$$

One notes from (1), (8), and (10) that

$$|\omega_0 - \omega_c| = \alpha\,|S(t)| = \alpha\,|H^*f| \leqq \alpha f_m \int_0^\infty H(\tau)\, d\tau = \alpha f_m\,.$$

Thus, the local oscillator cannot establish synchronism, i.e., $\omega_i = \omega_0$, if $\Delta\omega(t) = \omega_i - \omega_c > \alpha f_m$ $(t \geqq 0)$. Anticipating that $\Delta\omega/\alpha f_m \leqq 1$ in cases

which involve synchronism, we define the relative deviation $\Omega(t)$ by the relation

$$\Omega(t) = \frac{\Delta\omega(t)}{\alpha f_m}. \tag{13}$$

Equations (11), (12) and (13) can be combined to give

$$g\frac{dg}{dx} = \Omega(t) + \frac{1}{\sigma_1}\frac{d\Omega}{dt} - C(x) - kg + (1-\beta)kg\frac{dC}{dx}. \tag{14}$$

Also, by (4), (8), (9), (11), and (12),

$$g[x(0^+)] = \frac{k}{\pi\sigma_1}\left(\frac{d\epsilon}{dt}\right)_{0^+} = \frac{\Omega(0)}{k} \tag{15}$$

$$x(0^-) = x(0) = x(0^+) = 0. \tag{16}$$

To render the analysis of (14) to (16) tractable, we consider only specific classes of $\Omega(t)$. These deviations, although restricted, are chosen so that the associated synchronization behavior is representative of the more general phenomena. Accordingly, we discuss the following two classes:

(*i*) As a characterization of sudden input changes, $\Omega(t)$ is assumed to be a step function of magnitude ξ, causing either a temporary or permanent loss of synchronism. The maximum value of ξ for which the system returns to a synchronous state is denoted by ξ_+; namely,

$$\xi_+ = \sup\{\xi \mid \xi \geqq 0; \quad g(x) \to 0 \ (t \to \infty)\} \tag{17}$$

where

$$\Omega(t) = \begin{cases} \xi & (t \geqq 0) \\ 0 & (t < 0) \end{cases}.$$

Similarly, for negative values of ξ,

$$\xi_- = \inf\{\xi \mid \xi < 0; \qquad g(x) \to 0 \ (t \to \infty)\}. \tag{18}$$

We define the "relative" capture range ξ_c by the relation

$$\xi_c = \xi_+ - \xi_-. \tag{19}$$

(*ii*) As a generalization of the deviation type above, $\Omega(t)$ is assumed initially arbitrary but asymptotically such that

$$\Omega(t) \to \gamma = \text{const} \ (t \to \infty).$$

In a manner similar to that of (i), we define the relative "pull-in" range γ_p by the relation

$$\gamma_p = \gamma_+ - \gamma_- \tag{20}$$

where

$$\gamma_+ = \sup\{\gamma \mid \gamma \geqq 0; \quad g(x) \to 0, \Omega(t) \to \gamma\ (t \to \infty)\}$$
$$\gamma_- = \inf\{\gamma \mid \gamma < 0; \quad g(x) \to 0, \Omega(t) \to \gamma\ (t \to \infty)\}.$$

Since this latter class contains the class of (i), γ_p serves as a lower bound on ξ_c.

The capture and pull-in range as well as the steady-state phase error are significant as measures of the effectiveness of the system to correct for input deviations. This synchronizing capability is of prime importance; however, in the treatment here solutions $x(t)$ are not of direct interest.

III. PHASE PLANE REPRESENTATION

The phase plane is perhaps the most natural means of describing the behavior of the phase-controlled system. As shown in Fig. 2, the solution trajectories in the (g,x) plane assume two essential forms: Those approaching singular or critical points x_0 on the abscissa represent the capture phenomenon; those running on indefinitely, the noncapture or nonsynchronous state. The values of x_0 are determined directly from (11) and (12).[4,5] For $\Omega(t) = \xi\ (t \geqq 0)$,

$$C(x_0) = \xi \quad g(x_0) = 0. \tag{21}$$

Similarly, for $\Omega(t) \to \gamma\ (t \to \infty)$,

$$C(x_0) = \gamma, \qquad g(x_0) = 0 \quad (t \to \infty). \tag{22}$$

Since in both of these instances $|C(x_0)| \leqq 1$, a necessary condition for the presence of critical points and, consequently, capture and pull-in is that

$$|\xi| \leqq 1, \qquad |\gamma| \leqq 1. \tag{23}$$

As regards the nature of these points, it is shown in Appendix A that x_0 is either a stable or saddle point according as $(dC/dx)_{x_0}$ is positive or negative. Borderline cases, e.g., those involving the condition

$$(dC/dx)_{x_0} = 0$$

and the coalescence of stable and saddle points, are not of physical

interest as infinitesimal variations of the system parameters eliminate the borderline properties. In the event that $C(x)$ is discontinuous, the above criteria are applied to a regular sequence, i.e., a sequence $\{C_n(x)\}_0^\infty$ of continuous and differentiable functions for which

$$C_n(x) \to C(x) \ (n \to \infty),$$

the desired conditions obtained in the limit.

IV. TEST TRAJECTORIES — A VARIATIONAL APPROACH

At this point we describe a method with which to obtain upper and lower bounds on the relative capture range. With reference to the generalized phase plane portrait of Fig. 3, the capture and noncapture trajectories are regarded as terminating on two different sets of points M and N, respectively, and emanating from various regions of the initial state domain D, e.g., D_M and D_N. Terminal points in N represent points at infinity; those in M, critical points. Saddles are included in the latter group because a trajectory which approaches such a point involves an infinite amount of time and therefore satisfies capture conditions (17) to (19). Stated in the most general manner, the basic problem to be treated here is the determination of contours Γ_c, namely, boundaries separating regions in D from which either capture or noncapture trajectories originate. For $\Omega(t) = \xi$, domain D becomes the g axis and contours Γ_c degenerate to points.

As a technique for locating Γ_c, we first construct "test trajectories," i.e., functions $h(x)$, which run from D to a position somewhere between sets M and N. The test paths are selected so that if the solution paths are known to lie entirely on either side of the former, then the latter must terminate on points in the terminal set lying on the same side. It is then

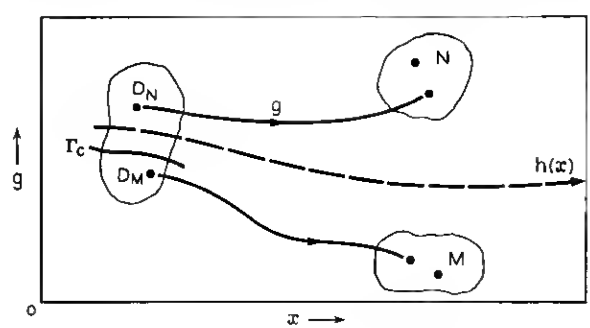

Fig. 3 — Generalized phase plane portrait.

possible to show by means of comparison theorems that for certain regions of the initial state domain positioned on one side of a test trajectory, the corresponding solution trajectories are similarly positioned. Consequently, the regions in D so determined are identified as either capture or noncapture domains forming bounds on Γ_c.

To make the above procedure precise, we utilize the following comparison theorem: Let $g(x(t))$ be a trajectory branch satisfying the autonomous equation

$$\frac{dg}{dx} = F(x,g) \quad (x \varepsilon I, t \varepsilon J)$$

where I denotes a closed interval of $x (x_a \leqq x \leqq x_b)$ and J, a branch interval of t defined by the inverse function $t = t(x)$ $(x \varepsilon I)$. If for a test trajectory $h(x)$,

$$\begin{gathered} \frac{dh}{dx} < (>)\ F(x,h) \quad (x \varepsilon I) \\ h(x_a) \leqq (\geqq)\ g(x_a) \\ h(x), g(x), t(x) \varepsilon C \quad (x \varepsilon I, t \varepsilon J) \end{gathered} \tag{24}$$

then

$$h(x) < (>)\ g(x), \qquad x_a < x \leqq x_b. \tag{25}$$

A brief proof of (24) and (25) is outlined in Appendix B. Although normally restricted to continuous functions, this theorem can be applied to generalized functions through the related regular sequences.

With $g(x_a)$ positioned in the initial state domain and the test trajectory constructed as a family of functions involving parameters $\lambda_1, \cdots, \lambda_n$, i.e., $h(x,\lambda)$, the inequalities of (24) can usually be reduced to tbe form

$$g(x_a) > (<)\ G(x,\lambda) \tag{26}$$

where the inequalities are sometimes reversed. Since the G function is known, there are obtained from (26) regions of domain D for which the resulting trajectories lie either above or below those of the test functions; if in the sense described above, sets M and N are isolated by $h(x,\lambda)$, then these regions constitute portions of the capture and noncapture domains to be determined. The optimal bounds on $g(x_a)$ for the given family $h(x,\lambda)$ are derived by extremalizing G with respect to x and λ:

$$g(x_a) \quad \begin{cases} > g_U = \inf_{\lambda} \cdot \sup_{x} G(x,\lambda) \\ < g_L = \sup_{\lambda} \cdot \inf_{x} G(x,\lambda) \end{cases}. \tag{27}$$

The quality of these bounds depends on that of the test function in that the closer $g(x)$ is approximated by $h(x)$, the better the contours Γ_c are delineated. Unlike the functions associated with Liapounoff's second method,[4,5] $h(x)$ is related directly to whatever properties of the actual trajectory are known and is constructed so as to conform as closely as necessary to these properties.

V. GENERAL SYMMETRIC COMPARATOR — RC FILTER

In this section we apply the method of test trajectories to phase-controlled systems employing RC filters and symmetric comparators, viz., those defined by the relations

$$C(x) = C(x + 2n) \quad (n = 0, \pm 1, \cdots,) \tag{28}$$

$$C(x) = -C(-x) \tag{29}$$

$$C(x) \geqq 0 \quad (0 \leqq x \leqq 1) \tag{30}$$

where in accordance with (3) to (5) and (6)

$$C(0) = 0 \tag{31}$$

$$C(x) \leqq 1 \quad (0 \leqq x \leqq 1). \tag{32}$$

For the RC filter $\sigma_2 \to \infty$ $(\beta = 1)$ in (14); therefore,

$$g \frac{dg}{dx} = \Omega + \frac{1}{\sigma_1} \frac{d\Omega}{dt} - C(x) - kg. \tag{33}$$

5.1 *Relative Capture Range*

With $\Omega(t) = \xi$, (33) reduces to

$$\frac{dg}{dx} = \frac{1}{g} [\xi - kg - C(x)] = F(x,g). \tag{34}$$

It is noted in this expression that trajectories corresponding to inputs ξ and $-\xi$ are symmetric about the origin of the phase plane in that the respective solutions $g_\xi(x)$ and $g_{-\xi}(x)$ are related by

$$g_\xi(x) = -g_{-\xi}(-x).$$

Consequently, both solutions exhibit the same capture characteristics, and

$$\xi_+ = -\xi_- \tag{35}$$

or

$$\xi_c = 2\xi_+ = -2\xi_- . \tag{36}$$

As a result, only positive input deviations need be considered. Referring to (15), one obtains for the initial state line

$$g(0) = \frac{\xi}{k} \geqq 0. \tag{37}$$

In order to establish crude noncapture intervals on this line, we employ the following test function:

$$h_1(x) = \sum_{n=0}^{\infty} \mu_1(x - 2n) \quad (0 \leqq x < \infty) \tag{38}$$

where

$$\mu_1(x) = \begin{cases} \dfrac{\lambda}{k}\left[1 - \dfrac{1}{\rho}\displaystyle\int_0^x C(z)\,dz\right]^{\frac{1}{2}} & (-1 \leqq x \leqq 1) \\ 0 & (|\,x\,| \geqq 1) \end{cases}$$

$$\rho = \int_0^1 C(z)\,dz.$$

The functional form chosen for $h_1(x)$, as well as most of those to follow, serves two basic aims:

(*i*) In addition to having the general appearance of a noncapture trajectory, $h_1(x)$ isolates the set of critical points on the x axis from the region above the test trajectory; i.e., solution trajectories which lie above $h_1(x)$ are bounded away from the x axis (cf. Section IV).

(*ii*) With respect to factor $\mu_1(x)$, $h_1(x)$ relates closely to the algebraic structure of (34).

We first verify property (*i*). By (28) to (32),

$$\rho = \sup_x \int_0^x C(z)\,dz\,.$$

Thus

$$\frac{1}{\rho}\int_0^x C(z)\,dz \leqq 1$$

and

$$\mu_1(x) \geqq 0$$

whence

$$h_1(x) \geqq 0 \quad (0 \leqq x \leqq \infty). \tag{39}$$

The comparison theorem applied to (34), (38) and (39) insures that

$$0 \leqq h_1(x) < g(x) \quad (x > 0)$$

for all values of ξ satisfying the inequalities

$$\begin{aligned} \xi > C(x) + kh_1 + h_1 \frac{dh_1}{dx} \\ = \left(1 - \frac{\lambda^2}{2\rho k^2}\right) C(x) + kh_1(x) = G(x,h_1) \quad (x \geqq 0) \end{aligned} \tag{40}$$

$$h_1(0) \leqq g(0) = \frac{\xi}{k}. \tag{41}$$

Since $C(0) = 0$, condition (41) is contained implicitly in (40), the latter expression corresponding to that of (26).

Functions $g(x)$ as trajectories bounded away from the x axis constitute noncapture solutions; hence, any ξ interval consistent with (40) is a noncapture interval. For obtaining rough limits, we set $\lambda^2 = 2\rho k^2$. Inequality (40) then becomes

$$\xi > kh_1(x) \quad (x \geqq 0).$$

Therefore, as indicated by (27) and (35), all values of ξ which satisfy the relation

$$\xi > \xi_1 = \sup_{x \geqq 0} kh_1(x) = \sup_{|x| \leqq 1} k\mu_1(x) = k(2\rho)^{\frac{1}{2}} \tag{42}$$

lie in the noncapture interval with the extremes $+\xi_1$ and $-\xi_1$ representing exterior bounds on the capture range. Although directly significant, these limits are derived primarily to provide essential information in more general calculations.

For a refined treatment we construct a more complex test function $h_2(x)$: Let $g_0(x)$ denote the separatrix solution which runs from the initial point $(g_0(0), 0)$ through the last saddle in the interval nearest to the point $x = 1$, and let ξ_0 denote the related input deviation. We then define

$$h_2(x) = \sum_{n=0}^{\infty} \mu_2(x - 2n) \quad (0 \leqq x < \infty) \tag{43}$$

where

$$\mu_2(x) = \begin{cases} \mu_1(x) & (\lambda = \xi_0, -1 \leqq x < 0) \\ g_0(x) & (0 \leqq x \leqq x_0) \\ 0 & (x_0 \leqq x \leqq 1) \end{cases}.$$

In addition to assuming the existence and uniqueness of $g_0(x)$, we suppose further that $g_0(x) \geqq 0$ $(0 \leqq x \leqq x_0)$, for if $g_0(x) = k/\sigma_1\, dx/dt < 0$,

then $x(t)$ decreases as time increases, and the trajectory is depicted receding from the point x_0 in opposition to the definition of $g_0(x)$; thus,

$$\mu_2(x) \geqq 0 \quad (-1 \leqq x \leqq 1)$$

and

$$h_2(x) \geqq 0 \quad (0 \leqq x < \infty). \tag{44}$$

Also,

$$h_2(0) = g_0(0) = \frac{\xi}{k}. \tag{45}$$

Therefore, we have the same sufficient requirements for noncapture values of ξ as those of (40) and (41):

$$\xi < C(x) + kh_2 + h_2 \frac{dh_2}{dx} = G(x,h_2) \tag{46}$$

$$h_2(0) = \frac{\xi_0}{k} \leqq g(0) = \frac{\xi}{k}. \tag{47}$$

To determine $\sup\limits_{x \geqq 0} G(x,h_2)$, we first derive several conditions relative to x_0, ξ_0, and g_0. With the selection of g_0 as a proper solution, (34), (46) and (47) combine to yield

$$G(x,g_0) = \xi_0 \quad (0 \leqq x \leqq x_0). \tag{48}$$

We next let x_m represent the greatest critical point value in the interval $x_0 < x \leqq 1$; viz.,

$$x_m = \sup_i \{x_i \mid C(x_i) = \xi_0; \qquad x_0 < x_i \leqq 1\}.$$

Inasmuch as x_0 is required by definition to be the only saddle in the interval $x_0 \leqq x \leqq 1$, x_m must be a stable point; then, $(dC/dx)_{x_m} > 0$, as discussed in Section III, with the result that $C(x) > C(x_m)$ for $x_m \leqq x \leqq 1$. However, since $C(x)$ as an odd function satisfying

$$C(1) = \tfrac{1}{2}\{C(1^+) + C(1^-)\} = C(0) = 0$$

must either approach zero or jump discontinuously to zero as $x \to 1$. x_m cannot exist; hence,

$$C(x) \neq \xi_0(x_0 < x \leqq 1), \quad C(x_0) = \xi_0$$

and

$$\left(\frac{dC}{dx}\right)_{x_0} < 0.$$

These conditions imply that

$$C(x) \leqq C(x_0) \quad (x_0 \leqq x \leqq 1)$$

whence

$$\sup_{x_0 \leqq x \leqq 1} G(x,0) = \sup C(x) = C(x_0) = \xi_0 . \tag{49}$$

In addition, it is noted that because of the infinite amount of time for the g_0 trajectory to reach the saddle point, ξ_0 lies within the capture range and outside that defined by inequality (42); i. e.,

$$\xi_0 \leqq \xi_1 = k(2\rho)^{\frac{1}{2}}.$$

Consequently,

$$\begin{aligned} \sup_{-1 \leqq x < 0} G(x,\mu_1) &= \sup \left[\left(1 - \frac{\xi_0^{\,2}}{2\rho k^2} \right) C(x) + k\mu_1(x) \right] \\ &= \sup \,[k\mu_1(x)] \\ &= \xi_0 . \end{aligned} \tag{50}$$

As a result of (48), (49) and (50), the noncapture range is given by

$$\xi > \sup_{x \geqq 0} G(x,h_2) = \sup_{|x| \leqq 1} G(x,\mu_2) = \xi_0 . \tag{51}$$

That values of ξ less than ξ_0 are in the capture range is shown by reversing the inequalities in (46) and (47) and invoking topological properties of the phase plane portrait: Values of ξ for which

$$\xi < G(x,g_0) = \xi_0 \quad (0 \leqq x \leqq x_0) \tag{52}$$

correspond to solutions $g(x)$ which satisfy the relation

$$g(x) < g_0(x) \tag{53}$$

a condition indicating that for such ξ the associated solution trajectories either terminate on stable nodes or cross the x axis at a point $x = \hat{x} < x_0$ and proceed in the $-x$ direction. As shown by the phase portrait of Fig. 4, the crossover point $\hat{C}$ in the latter case must be located so that a saddle S lies just to the right and a focus F just to the left, for the trajectory would otherwise intersect either itself or a separatrix of S. Using the same argument proves that $g(x)$ must in addition spiral in toward F. Consequently, (52) is a capture condition and $\pm\xi_0$ are exact exterior bounds on the capture range. By (23) and (36)

$$\xi_c = 2\xi_0 \leqq 2 \tag{54}$$

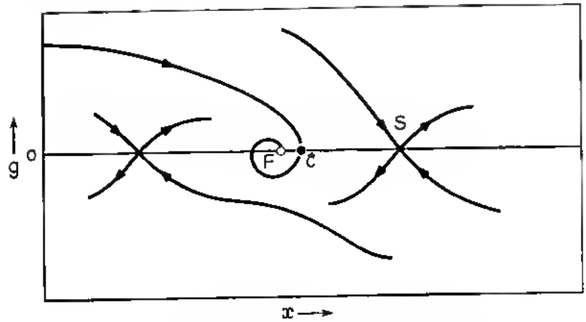

Fig. 4 — Phase portrait of trajectories.

or

$$\xi_+ = \xi_0 \leqq 1. \tag{55}$$

Two significant results have so far been obtained:

(i) The possibility of the system's reaching a synchronous state in a region outside the first domain does not exist; i.e., the steady state phase error x_{ss} must be such that

$$x_{ss} \epsilon \left\{ x_i \mid \xi = C(x_i); \quad \left(\frac{dC}{dx}\right)_{x_i} > 0; \quad 0 \leqq x_i \leqq 1 \right\}. \tag{56}$$

(ii) The determination of the capture range depends solely on that of the separatrix solution $g_0(x)$.

Among the set of stable points in (56) the appropriate x_i, namely, the exact value of x_{ss}, can be found by the application of the above procedure to each separatrix spanning the intervals $0 \leqq \xi/k \leqq \xi_0/k$ and $0 \leqq x \leqq x_0$ on the initial state line and x axis, respectively. For example, the first stable point $x_1 = \inf \{x_i\}$ is isolated by the separatrix which connects the points $\hat{\xi}/k$ and

$$\hat{x} = \inf \left\{ x_j \mid \hat{\xi} = C(x_j); \quad \left(\frac{dC}{dx}\right)_{x_j} < 0; \quad 0 \leqq x_j \leqq x_0 \right\},$$

in that the solutions for all $\xi < \hat{\xi}$ ($\xi > 0$) lie below this separatrix and $x_{ss} = x_1$. Each x_i can be handled in this manner.

In general, g_0 cannot be determined exactly; however, as regards specific comparators, graphical methods are found to yield results as accurate as one desires because of the relatively small distance traversed by the associated trajectory. The use of such graphical approximations for obtaining the capture range is prevalent in the literature in advance of any formal justification. It is important to point out that even for

the RC filter case, interior and exterior bounds are sometimes more convenient and useful than the graphical solutions.

5.2 *Asymptotic Results for Small and Large k*

It is possible in the case of vanishing k to derive explicit relations for both the capture range and $g_0(x)$. We first consider an exterior bound formulation based on the test function

$$h_3(x) = \sum_{n=0}^{\infty} \mu_3(x - 2n) \quad (x \geqq 0) \tag{57}$$

where

$$\mu_3(x) = \begin{cases} \left[2\rho - 2\int_0^x C(z)\,dz\right]^{\frac{1}{2}} & (-1 \leqq x \leqq x) \\ 0 \quad (|x| \geqq 0). \end{cases}$$

Note that

$$\mu_3(1) = 0$$
$$\mu_3(x) > 0 \quad (-1 < x < 1)$$
$$h_3(x) \geqq 0 \quad (x \geqq 0)$$
$$h_3(0) = (2\rho)^{\frac{1}{2}}$$

in accordance with noncapture criteria (cf. (40) et seq.). Hence, as in (40), (41) and (42), exterior bounds $\pm\xi_2$ are given by the expression

$$\begin{aligned} \xi > \xi_2 &= \sup_{x \geqq 0} G(x,h_3) = \sup_{|x| \leqq 1} G(x,\mu_3) \\ &= \sup_{|x| \leqq 1} \left[2\rho k^2 - 2k^2 \int_0^x C(z)\,dz\right]^{\frac{1}{2}} \\ &= k(2\rho)^{\frac{1}{2}} \end{aligned} \tag{58}$$

which is consistent with the initial condition of (41); viz.,

$$h_3(0) = (2\rho)^{\frac{1}{2}} \leqq g(0) = \frac{\xi}{k}.$$

To treat the interior bound calculation, we use

$$h_4(x) = \left[\frac{\lambda^2}{k^2} + 2\lambda x - 2\int_0^x C(z)\,dz\right]^{\frac{1}{2}} \quad (0 \leqq x \leqq \bar{x} \leqq 1) \tag{59}$$

where

$$h_4(\bar{x}) = 0$$

$$h_4^2(x) > 0 \quad (0 \leqq x < \bar{x} \leqq 1)$$

and λ equals the largest positive value for which the quantity in the above brackets has one zero in x in the interval $0 \leqq x \leqq 1$. Such a value must exist since $\int_0^x C(z)dz$ $(x \leqq x \leqq 1)$ is monotonic increasing. In addition, for this zero condition to hold independently of k, $\lambda \to k(2\rho)^{\frac{1}{2}}$ and $\bar{x} \to 1$ as $k \to 0$; thus,

$$h_4(x) \underset{k \to 0}{\longrightarrow} \left[2\rho - 2 \int_0^x C(z)\, dz\right]^{\frac{1}{2}} \quad (0 \leqq x \leqq \bar{x}). \tag{60}$$

Test function $h_4(x)$ is seen to satisfy the same requirements as those discussed in connection with (52) through (55). Accordingly, interior bounds $\pm\xi_3$ are given by the relation

$$\begin{aligned} \xi < \xi_3 &= \inf_{x<\bar{x}} G(x,h_4) = \inf \left\{\lambda + k \left[\frac{\lambda^2}{k^2} + 2\lambda x - 2 \int_0^x C(z)\, dz\right]^{\frac{1}{2}}\right\} \\ &= \lambda \underset{k \to 0}{\longrightarrow} k(2\rho)^{\frac{1}{2}} = \xi_2 \end{aligned} \tag{61}$$

which is also consistent with the initial condition

$$h_4(0) = \frac{\lambda}{k} \geqq g(0) = \frac{\xi}{k}.$$

Inasmuch as $\xi_3 \leqq \xi_+ \leqq \xi_2$ and $\xi_3 \to \xi_2(k \to 0)$, the results of (58), (60), and (61) yield

$$\begin{aligned} \xi_+ &= \frac{\xi_c}{2} \sim k(2\rho)^{\frac{1}{2}} && (k \to 0) \\ g_0(x) &\sim \left[2\rho - 2 \int_0^x C(z)\, dz\right]^{\frac{1}{2}} && (k \to 0) \end{aligned} \tag{62}$$

where

$$\rho = \int_0^1 C(z)\, dz.$$

Therefore, the choice of asymptotically equal test functions results in coalescent bounds.

In considering large values of k, we take

$$h_5(x) = \hat{x} + \lambda - x \quad (0 \leqq x \leqq \hat{x} + \lambda < 1) \tag{63}$$

where

$$0 \leqq C(x) < 1 \quad (0 \leqq x < \hat{x} \leqq 1)$$
$$C(\hat{x} + \lambda) \underset{\lambda \to 0}{\longrightarrow} 1$$
$$\hat{x} > 0 \qquad \hat{x} + \lambda > 0$$

and $\lambda \gtreqless 0$ according as $(dC/dx)_{\hat{x}} \gtreqless 0$. Clearly, λ is employed only in the event that $C(x)$ has a discontinuity at $x = \hat{x}$. Since the values of ξ specified by

$$\xi < G(x,h_5) = h_5 \frac{dh_5}{dx} + 2kh_5 + C(x)$$

also satisfy the basic initial criterion

$$kg(0) = \xi < G(0,h_5(0)) = -(\hat{x} + \lambda) + kh_5(0) \leqq kh_5(0)$$

interior bounds $\pm\xi_4$ are given by

$$\xi < \xi_4 = \inf_{x < \hat{x}+\lambda} G(x,h_5) = \inf\,[(k - 1)(\hat{x} + \lambda - x) + C(x)]. \tag{64}$$

Employing the limits

$$G(x,h_5) \underset{k \to \infty}{\longrightarrow} \infty \quad (x < \hat{x} + \lambda)$$
$$G[(\hat{x} + \lambda), h_5(\hat{x} + \lambda)] = C(\hat{x} + \lambda)$$

yields

$$\lim_{\lambda \to 0} \lim_{k \to \infty} \xi_4 = \lim_{\lambda \to 0} C(\hat{x} + \lambda) = C(\hat{x}) = 1 \leqq \xi_+ .$$

However, according to (23), $\xi_+ \leqq 1$; therefore,

$$\xi_+ = \frac{\xi_c}{2} \underset{k \to \infty}{\longrightarrow} 1. \tag{65}$$

One can show in a similar manner that if the system captures for a given ξ and k, a capture condition results for all smaller values of k. Consequently, the relative capture as a function of k^{-2} assumes the general form depicted in Fig. 5 with the asymptotic behavior given by (62) and (65).

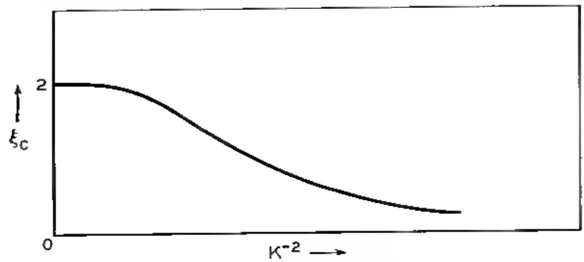

Fig. 5 — General form of relative capture as a function of k^{-2}.

VI. SAWTOOTH COMPARATOR — RC FILTER

Directly related to the formulation and results of Section V is the RC filter-sawtooth comparator system in which

$$C(x) = \sum_{n=-\infty}^{\infty} C_0(x - 2n) \quad (-\infty < x < \infty)$$

$$C_0(x) = \begin{cases} x \ (-1 < x < 1) \\ 0 \ |x| \geqq 1 \end{cases} \tag{66}$$

$$f_m = \pi, \qquad k = \left(\frac{\sigma_1}{\alpha}\right)^{\frac{1}{2}}, \qquad \beta = 1.$$

The relatively simple form assumed by $C(x)$ in the first domain enables one to derive exact expressions for both $g_0(x)$ and ξ_0. It is shown in Appendix C (cf. (99)) that

$$\xi_0 = \xi_+ = \frac{\xi_c}{2} = \begin{cases} \dfrac{k}{k + \exp\left[\dfrac{k}{\tau_0}\left(\tan^{-1}\dfrac{\tau_0}{k} - \pi\right)\right]} & (k < 2) \\ 1 \quad (k \geqq 2) \end{cases} \tag{67}$$

where $\tau_0^2 = (4 - k^2)$ and $0 < \tan^{-1}(\cdot) < \pi/2$. For small k, (67) can be expressed as

$$\xi_0 \sim 2k = 2\sqrt{\frac{\sigma_1}{\alpha}} \quad \left(k, \frac{\sigma_1}{\alpha} \to 0\right) \tag{68}$$

which checks with (62). In addition, from (20 et seq.)

$$\gamma_p \leqq \xi_c . \tag{69}$$

An exact derivation of γ_p valid for the sawtooth and general filter case has been obtained by A. J. Goldstein.[6]

As regards steady state phase error, relations (21) and (56) give

$$x_{ss} = C^{-1}(\xi_0)\,|_{|x|\leqq 1} = \frac{\xi_c}{2}. \tag{70}$$

VII. SAWTOOTH COMPARATOR — LAG FILTER

The capture results as developed in Section V do not apply completely to the lag filter; however, the calculation of exterior and interior bounds in this case follows closely that of (40), (41) and (52). For $\Omega(t) = \xi$, $\beta > 1$, and $C(x)$ specified by (66), (14) reduces to

$$\begin{aligned}\xi &= C(x) + \beta kg + g\frac{dg}{dx} - 2(\beta - 1)kg \sum_n \delta\{x - (2n + 1)\} \\ &= G_1(x,g)\end{aligned} \tag{71}$$

where

$$k = \left(\frac{\sigma_1}{\alpha}\right)^{\frac{1}{2}}$$

and

$$\beta = 1 + \frac{\alpha}{\sigma_2}.$$

Using test function $h_1(x)$ of (38) yields the following noncapture values of ξ (cf. (40)):

$$\xi > G_1(x,h_1)(x \geqq 0)$$

where, as required, $kg(0) = \xi > k\beta h_1(0) > kh_1(0)$. Hence, exterior bounds $\pm\xi_U$ are given by

$$\begin{aligned}\xi_U &= \sup_{x\geqq 0,\lambda\geqq 0} G_1(x,h_1) = \sup_{|x|\leqq 1,\lambda\geqq 0} G_1(x,\mu_1) \\ &= \begin{cases}\dfrac{\beta k}{2}(4 - \beta^2k^2)^{\frac{1}{2}} & (\beta k < \sqrt{2}) \\ 1 & (\beta k \geqq \sqrt{2}).\end{cases}\end{aligned} \tag{72}$$

Using separatrix solution $g_L(x)$ of Appendix C as a test function yields the following capture values of ξ (cf. (52) and (98) et seq.):

$$\xi < G_1(x,g_L) = \xi_L \quad (0 \leqq x \leqq 1) \tag{73}$$

where

$$\xi_L = \begin{cases} \dfrac{k}{k + [1 - k^2(\beta - 1)]^{\frac{1}{2}} \exp\left\{\dfrac{\beta k}{\tau_1}\left[\tan^{-1}\dfrac{\tau_1}{(2-\beta)k} - \pi\right]\right\}} & (\beta k < 2) \\ 1 & (\beta k \geqq 2) \end{cases}$$

$\tau_1^2 = (4 - \beta^2 k^2)$

and

$$0 < \tan^{-1}(\cdot) < \pi.$$

Therefore,

$$\xi_L \leqq \frac{\xi_c}{2} \leqq \xi_U. \tag{74}$$

With regard to small k, the relations for ξ_U and ξ_L become

$$\xi_U \sim \beta k \quad (k \to 0) \tag{75}$$

$$\xi_L \sim k \quad (k \to 0). \tag{76}$$

As reflected by parameter β, the spread between ξ_U and ξ_L for small k appears significant, but in most cases involving small k, $\sigma_2 \gg \alpha$ and $\beta \cong 1$. Under more general conditions these bounds can be refined by constructing more complex test functions.

According to the discussion in Section II (cf. (20) et seq.), ξ_U serves also as an upper bound on the pull-in range; viz.,

$$\gamma_p = 2\gamma_+ \leqq 2\xi_U. \tag{77}$$

VIII. SINE COMPARATOR — RC FILTER

The sine-RC system is perhaps the most important practically, having received a great deal of attention in the literature. Several investigators have obtained capture results based on phase plane constructions of $g_0(x)$ for large and medium values of k.[1,7] For small k, approximate results have been derived by both analytic and experimental means, although these approximations are found to differ somewhat.[1,2] In this section we discuss the exact asymptotic form of the relative capture range for small k.

The normalized sine comparator is represented by

$$C(x) = \sin(\pi x) \quad (-\infty < x < \infty) \tag{78}$$

whence

$$\rho = \int_0^1 C(x)\,dx = \frac{2}{\pi}$$

$$f_m = 1$$

and

$$k = \left(\frac{\pi\sigma_1}{\alpha}\right)^{\frac{1}{2}}.$$

Consequently, the first relation of (62) can be written as

$$\xi_+ = \frac{\xi_c}{2} \sim \frac{2}{\sqrt{\pi}}\,k = 2\sqrt{\frac{\sigma_1}{\alpha}} \quad \left(k, \frac{\sigma_1}{\alpha} \to 0\right) \tag{79}$$

or

$$\xi_+ \sqrt{\frac{\alpha}{\sigma_1}} \to 2 \quad \left(\frac{\sigma_1}{\alpha} \to 0\right). \tag{80}$$

The numerical limit in (80) has been estimated by McAleer[2] and Jelonek[1] to be I.00 and 1.27, respectively. As lower bounds, these approximations differ appreciably.

IX. SUMMARY

The more significant results of the above development can be outlined as follows:

(*i*) Tbe variational formulation given by (27), a generalization of which allows one to treat similar problems in n-dimensional phase space.

(*ii*) Bounds given by (42) and (61), yielding respectively noncapture and capture ranges for the general symmetric comparator and the simple RC filter.

(*iii*) The first domain phenomenon associated with the symmetric-RC case of (*ii*) (cf. (56) et seq.).

(*iv*) The exact asymptotic capture relations given by (62) for the symmetric-RC case.

(*v*) The pull-in criteria given by (20 et seq.), (42), (61), and (62) for the symmetric-lag and RC cases.

(*vi*) The specific capture relations given by Sections VI through VIII for the sawtooth-sine and lag-RC cases.

(*vii*) The asymptotic relation given by (80) for small k, simple RC filters, and sine comparators.

APPENDIX A

Critical Point Type

To determine the behavior of trajectories near a critical point

$$(x = x_0, \qquad g = 0),$$

we linearize (11) and (12) about this point and investigate the natural modes of the corresponding linear solutions: As shown by Liapounoff,[5] these modes identify the type of critical point in that the presence of at least one exponentially increasing component indicates a saddle point, and the presence of only decreasing components, a stable (nodal or focal) point. Equations (11) and (12) are linearized by expanding the right-hand sides in a double Taylor series in x and g about the point

$$(x = x_0, \qquad g = 0),$$

and dropping all terms beyond the first order. Accordingly, we obtain from (11), (12), (21) and (22)

$$\begin{aligned} \frac{dx}{dt} &= ag \\ \frac{dg}{dt} &= b_1 x + b_2 g \end{aligned} \tag{81}$$

where

$$\begin{aligned} a &= \frac{\sigma_1}{k} \\ b_1 &= -\frac{\sigma_1}{k}\left(\frac{dC}{dx}\right)_{x_0} \\ b_2 &= \sigma_1\left[(1-\beta)\left(\frac{dC}{dx}\right)_{x_0} - 1\right]. \end{aligned}$$

Equations (81) combine to yield

$$\frac{d^2x}{dt^2} - b_2\frac{dx}{dt} - ab_1 x \tag{82}$$

the exponential components of the related solutions increasing or decreasing according as the roots of the secular equation

$$\lambda^2 - b_2\lambda - ab_1 = 0 \tag{83}$$

are positive or negative. Moreover, if $(dC/dx)_{x_0} > 0$, then $b_1 < 0$, $b_2 < 0$, and

$$\lambda = \frac{b_2}{2} \pm \left[\frac{b_2^{\,2}}{4} + ab_1\right]^{\frac{1}{2}} < 0. \tag{84}$$

Also, if $dC/dx_{x_0} < 0$, then $b_1 > 0$ and at least one value of λ given by (84) is positive; i. e.,

$$\lambda = \frac{b_2}{2} + \left[\frac{b_2^{\,2}}{4} + ab_1\right]^{\frac{1}{2}} > 0.$$

Hence, the critical point in question is stable provided $(dC/dx)_{x_0} > 0$ and a saddle provided $(dC/dx)_{x_0} < 0$.

APPENDIX B

Comparison Theorem

Theorem

If for a branch trajectory $g(x(t))$ and a test trajectory $h(x)$,

$$\frac{dg}{dx} = F(x,g) \quad (x \; \varepsilon \; I, t \; \varepsilon \; J)$$

$$\frac{dh}{dx} < (>) F(x,h) \quad (x \; \varepsilon \; I)$$

$$h(x_a) \leqq (\geqq) g(x_a)$$

$$h(x), \qquad g(x), \qquad t(x) \; \varepsilon \; C \qquad (x \; \varepsilon \; I, t \; \varepsilon \; J), \tag{85}$$

where I denotes a closed interval of $x(x_a \leqq x \leqq x_b)$ and J, a branch interval of t defined by the inverse function $t = t(x)$ $(x \; \varepsilon \; I)$, then

$$h(x) < (>) g(x) \quad (x_a < x \leqq x_b). \tag{86}$$

Proof

Assume first that $h(x_a) < g(x_a)$ and $dh/dx < F(x,h)$ $(x \; \varepsilon \; I)$. With h and g continuous, either $h(x) < g(x)(x \; \varepsilon \; I)$ or the two trajectories intersect at some point $\bar{x}$ in the interval, i.e., $h(\bar{x}) = g(\bar{x})(x_a < \bar{x} \leqq x_b)$; however, an intersection implies that $(dh/dx)_{\bar{x}} \geqq (dg/dx)_{\bar{x}}$, a condition which contradicts the combined relation

$$\left(\frac{dh}{dx}\right)_{\bar{x}} < F(\bar{x},h(\bar{x})) = F(\bar{x},g(\bar{x})) = \left(\frac{dg}{dx}\right)_{\bar{x}}. \tag{87}$$

Therefore, in this case,

$$h(x) < g(x) \quad (x \varepsilon I).$$

If

$$h(x_a) = g(x_a)$$

and

$$\frac{dh}{dx} < F(x,h) \quad (x \varepsilon I)$$

$$\left(\frac{dh}{dx}\right)_{x_a} < F(x_a, h(x_a)) = F(x_a, g(x_a)) = \left(\frac{dg}{dx}\right)_{x_a}.$$

Consequently, $h(x) < g(x)$ over some portion of I. Ruling out additional intersections by (87), we obtain the general result

$$h(x) < g(x) \quad (x_a < x \leqq x_b).$$

The alternative set of inequalities is shown to be valid in a similar manner.

APPENDIX C

First Domain Trajectories in the Sawtooth Case

For treating the sawtooth comparator-lag filter case, we seek here solution trajectories of (71) which span the initial state line and saddle points in the first domain, viz., $0 \leqq x \leqq 1$. As defined in (66),

$$C(x) = \begin{cases} x & (0 \leqq x < 1) \\ 0 & (x = 1). \end{cases} \tag{88}$$

Hence, by (21 et seq.) the only saddle present in the first domain is that at the point $(x = 1, g = 0)$. We intend, therefore, to determine a solution $g_L(x)$ spanning the two points $(0,\xi/k)$ and $(1,0)$. It is noted that in the simple RC case, $g_L(x)$ is identical to $g_0(x)$ and ξ_L, to ξ_0 (cf. (43) and preceding).

Restricted to the interval $(x \leqq x \leqq 1)$ and solution $g_L(x)$, (71) becomes

$$\xi_L = x + \beta k g_L + g_L \frac{dg_L}{dx} - 2(\beta - 1)k g_L \delta(x - 1) \quad (0 \leqq x \leqq 1). \tag{89}$$

However, since $g_L(x)$ is required to vanish as $x \to 1$, $g_L\delta(x-1) = 0$; consequently,

$$\xi_L = x + \beta k g_L + g_L \frac{dg_L}{dx}. \tag{90}$$

Setting $z = x - \xi_L$ and $v = g_L/z$ reduces (90) to the form

$$\begin{aligned} 2\frac{dz}{z} &= -\frac{2v\,dv}{v^2 + \beta k v + 1} \\ &= \frac{\beta k}{\left(v + \frac{\beta k}{2}\right)^2 + \frac{\tau_1^{\,2}}{4}} - \frac{2v + \beta k}{v^2 + \beta k v + 1} \end{aligned} \tag{91}$$

where

$$\tau_1^{\,2} = (4 - \beta^2 k^2).$$

Integrating, we have

$$\ln\left[g_L^{\,2} + \beta k(x - \xi_L)g_L + (x - \xi_L)^2\right] = \frac{2\beta k}{\tau_1}\tan^{-1}\left[\frac{2g_L + \beta k(x - \xi_L)}{\tau_1(x - \xi_L)}\right] + C_3 \tag{92}$$

where C_3 = const. At the saddle point ($g_L = 0$, $x = 1$), (92) becomes

$$\ln\,(1 - \xi_L)^2 = \frac{2\beta k}{\tau_1}\tan^{-1}\left(\frac{\beta k}{\tau_1}\right) + C_3. \tag{93}$$

Therefore, g_L is given implicitly by the relation

$$\frac{g_L^{\,2} + \beta k(x - \xi_L)g_L + (x - \xi_L)^2}{(1 - \xi_L)^2} = \exp\left\{\frac{2\beta k}{\tau_1}\left[\tan^{-1}\left(\frac{2g_L + \beta k(x - \xi_L)}{\tau_1(x - \xi_L)}\right) - \tan^{-1}\left(\frac{\beta k}{\tau_1}\right)\right]\right\}. \tag{94}$$

An explicit expression for ξ_L is derived by inserting the initial condition $g_L(0) = \xi_L/k$ in (94); i. e.,

$$\frac{\xi_L^{\,2}}{k^2}\left[\frac{1 - k^2(\beta - 1)}{(1 - \xi_L)^2}\right] = \exp\left\{\frac{2\beta k}{\tau_1}\left[\tan^{-1}\left(\frac{\beta k^2 - 2}{\tau_1 k}\right) - \tan^{-1}\left(\frac{\beta k}{\tau_1}\right)\right]\right\}. \tag{95}$$

For $\xi_L \leqq 1$ (cf. 23) and $\beta k < 2$ (cf. 91), the argument of the first arc-tan function in (94) takes on an infinite value at only one point in the interval $(0 \leqq x \leqq 1)$; i. e.,

$$\frac{2g_L + \beta k(x - \xi_L)}{\tau_1(x - \xi_L)} \rightarrow \begin{cases} \dfrac{\beta k}{\tau_1} \geqq 0 & (x \rightarrow 1) \\ +\infty & (x \rightarrow \xi_L^{+}) \\ -\infty & (x \rightarrow \xi_L^{-}) \\ \left(\dfrac{\beta k^2 - 2}{\tau_1 k}\right) \lesseqgtr 0 & (x \rightarrow 0). \end{cases}$$

Thus, the arc-tan difference in (95) must be such that

$$0 < \tan^{-1}\left(\frac{\beta k^2 - 2}{\tau_1 k}\right) - \tan^{-1}\left(\frac{\beta k}{\tau_1}\right) = \tan^{-1}\left[\frac{-\tau_1}{(2 - \beta)k}\right] < \pi. \quad (96)$$

Equation (95) can then be written as

$$\frac{\xi_L[1 - k^2(\beta - 1)]^{\frac{1}{2}}}{k(1 - \xi_L)} = \exp\left\{\frac{\beta k}{\tau_1}\left[\tan^{-1}\left(\frac{-\tau_1}{(2 - \beta)k}\right)\right]\right\} \quad (97)$$

or

$$\xi_L = \frac{k}{k + [1 - k^2(\beta - 1)]^{\frac{1}{2}} \exp\left\{\dfrac{\beta k}{\tau_1}\left[\tan^{-1}\left(\dfrac{\tau_1}{(2 - \beta)k}\right) - \pi\right]\right\}} \quad (98)$$

$$(\beta k < 2)$$

where $0 < \tan^{-1}(\cdot) < \pi$. That $\xi_L = 1$ for $\beta k \geqq 2$ is shown by considering two values of βk, say $\beta k_1 \rightarrow 2$ and $\beta k_2 > 2$, and two trajectories with respective initial conditions $\xi_1/k_1 = 1/k_1$ and $\xi_2/k_2 = 1/k_2$. Since from (98), $\xi_L \rightarrow 1 = \xi_1 = \xi_2$ as $\beta k_1 \rightarrow 2$, the former trajectory is a capture type. However, $1/k_1 > 1/k_2$; therefore, the latter initial point and trajectory lie below those of the former. This implies that relations $\xi_2 = 1$ and $\beta k_2 > 2$ correspond to capture conditions. On the other hand, if $\xi_2 > 1$, noncapture trajectories result. Hence, $\xi_L = \xi_2 = 1$ for $\beta k > 2$.

Finally, capture criteria in the simple RC case are given by

$$\xi_L \xrightarrow[\beta \rightarrow 1]{} \xi_0 = \begin{cases} \dfrac{k}{k + \exp\left[\dfrac{k}{\tau_0}\left(\tan^{-1}\dfrac{\tau_0}{k} - \pi\right)\right]} & (k < 2) \\ 1 & (k \geqq 2) \end{cases} \quad (99)$$

where

$$\tau_0^2 = (4 - k^2)$$

and

$$0 < \tan^{-1}(\cdot) < \pi/2.$$

APPENDIX D

List of Symbols

φ_i — input phase

φ_0 — output phase

$\epsilon = \varphi_i - \varphi_0$ — phase error

$x = \dfrac{\epsilon}{\pi}$ — normalized phase error

x_{ss} — normalized steady-state phase error in the synchronous state

$f(\epsilon)$ — comparator output

f_m — $\sup\limits_{\epsilon} | f(\epsilon) |$

$C(x) = \dfrac{f(\pi x)}{f_m}$ — normalized comparator function

$\rho = \displaystyle\int_0^1 C(z)\, dz$

ω_c — free-running frequency of the local oscillator

$\Delta\omega(t)$ — input frequency deviation

$\Omega(t) = \dfrac{\Delta\omega(t)}{\alpha f_m}$ — normalized input frequency deviation

$C_1 = \text{const.}$ — magnitude of input frequency step

$\xi = \dfrac{C_1}{\alpha f_m}$ — relative input frequency step

ξ_+, ξ_- — relative positive and negative capture ranges, respectively

$\xi_c = \xi_+ - \xi_-$ — relative capture range

ξ_0 — relative input step corresponding to separatrix $g_0(x)$

$C_2 =$ const. — magnitude of asymptotic input frequency step

$\gamma = \dfrac{C_2}{\alpha f_m}$ — relative asymptotic input frequency step

γ_+, γ_- — relative positive and negative pull-in ranges, respectively

$\gamma_p = \gamma_+ - \gamma_-$ — relative pull-in range

$H(t)$ — filter impulse response

$S(t)$ — filter output

$$g = \frac{k}{\sigma_1}\frac{dx}{dt}$$

$g_0(x)$ — primary first domain separatrix, as defined by the description prior to (43)

$\delta(t)$ — Dirac delta function

α — gain constant; viz., $\dfrac{d\varphi_o}{dt} = \omega_c + \alpha S(t)$

φ_1, σ_2 — filter parameters (cf. Fig. 1(b))

$$\left.\begin{aligned} k &= \left(\frac{\pi\sigma_1}{\alpha f_m}\right)^{\frac{1}{2}} \\ \beta &= 1 + \frac{\alpha f_m}{\pi\sigma_2} \end{aligned}\right\} \text{— normalized system parameters}$$

$*$ — convolution

s — Laplace transform variable; i.e., $\mathfrak{L}\cdot f(t) = F(s)$

REFERENCES

1. Jelonek, Z., Celinski, O., and Syski, R., *Pulling Effect in Synchronized Systems*, Monograph No. 79, The Institution of Electrical Engineers, 1953.
2. Mc Aleer, H. T., Proc. I.R.E., **47**, 1959, p. 1137.
3. Temple, G., Proc. Roy. Soc. (London), **A228**, 1955, p. 175.
4. Stoker, J. J., *Nonlinear Vibrations*, Interscience Publishers, Inc., New York, 1950.
5. Minorsky, N., "The Theory of Oscillations," in *Dynamics and Nonlinear Mechanics*, Vol. 2, John Wiley & Sons, Inc., New York, 1958.
6. Goldstein, A. J., to be published.
7. Celinski, O., *Investigation of Synchronization in Phase Lock Oscillators*, unpublished Diploma Thesis, Polish University College, London, 1950.